Novel correlations in two dimensions: Two-body problem

1997 J. Phys. A: Math. Gen. 30 2557

(http://iopscience.iop.org/0305-4470/30/7/032)

J. Phys. A: Math. Gen. **30** (1997) 2557–2576. Printed in the UK PII: S0305-4470(97)79009-7

Novel correlations in two dimensions: Two-body problem

R K Bhaduri†, Avinash Khare‡, J Law§, M V N Murthy‖ and Diptiman Sen†‖

† Department of Physics and Astronomy, McMaster University, Hamilton L8S 4M1, Ontario, Canada

‡ Institute of Physics, Bhubaneswar 751 005, India

§ Department of Physics, University of Guelph, Guelph N1G 2W1, Ontario, Canada

‖ The Institute of Mathematical Sciences, Madras 600 113, India

Received 25 October 1996

Abstract. We discuss a many-body Hamiltonian with two- and three-body interactions in two dimensions introduced recently by Murthy, Bhaduri and Sen. Apart from an analysis of some exact solutions in the many-body system, we analyse in detail the two-body problem which is completely solvable. We show that the solution of the two-body problem reduces to solving a known differential equation due to Heun. We show that the two-body spectrum becomes remarkably simple for large interaction strengths and the level structure resembles that of the Landau levels. We also clarify the 'ultraviolet' regularization which is needed to define an inverse-square potential properly and discuss its implications for our model.

1. Introduction

Recently, there has been a revival of interest in the area of exactly solvable models in one and higher dimensions. A celebrated example of a solvable many-body system is the well known Calogero–Sutherland model (CSM) in one dimension [1–3]. The model has found a wide application in areas as diverse as quantum chaos and fractional statistics. The particles in the CSM are confined in a one-body oscillator potential or on the rim of a circle, and interact with each other through a two-body potential which varies as the inverse square of the distance between particles. The CSM and its variants in one dimension, such as the Haldane–Shastry model for spin chains [4, 5], have provided the paradigms to analyse more complicated interacting systems. A characteristic feature of the CSM is the structure of the highly correlated wavefunction. The correlations are built into the exact wavefunction through a Jastrow factor $(x_i - x_j)^{\lambda}|x_i - x_j|^{\alpha}$ for any pair of particles denoted by i, j. The exponents on the correlator are related to the strength of the inverse-square interaction. Notice that this factor is asymmetric (symmetric) in particle labels for $\lambda = 1(0)$ and vanishes as the two particles approach each other. A generalization of this in two dimensions is to be found in Laughlin's trial wavefunction [6] where the correlations are built in through the factor $(z_i - z_j)$, where z_i are the particle coordinates in complex notation. The corresponding Hamiltonian for which the Laughlin wavefunction is an exact ground state has not been analysed to the same degree of detail as the CSM. It is known that the ground state for a Hamiltonian describes spin polarized electrons in the lowest Landau level with a short-range repulsive interaction [7]. It is also known that such correlations

‖ Permanent address: Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, India.

0305-4470/97/072557+20$19.50

are present in the exact ground state of a spin Hamiltonian [8] in two dimensions. The anyon Hamiltonian [9, 10] in two-space dimensions is another example where the Jastrow correlation appears [11]. While the two-anyon problem is exactly solvable, the many-anyon problem is not. For a system of anyons confined in an oscillator potential many exact solutions and their properties are known but, unlike the CSM in one dimension, the analytical solution of the full many-body problem is not tractable [12]. It is therefore of great interest to find models analogous to the CSM in higher dimensions.

In a recent paper [13], three of us proposed a model in two-space dimensions with nontrivial two- and three-body interactions which could be solved exactly for ground states and some excited states. It betrayed some similarity to both CSM in one dimension and the anyonic model in two dimensions through the spectrum. The model was devised by noting that in two dimensions another form of the pair correlator exists with which a Jastrow-type many-body wavefunction may be constructed, namely

$$X_{ij} = x_i y_j - x_j y_i. \tag{1}$$

The correlation is, by definition, asymmetric and goes to zero as two particles approach each other. In addition, it introduces zeros in the wavefunction whenever the relative angle between the two particles goes to zero or π. The difference with the Jastrow–Laughlin form is also significant; X_{ij} in (1) is a pseudo-scalar. Unlike the Laughlin type of correlation, it does not impart any angular momentum to the two-dimensional wavefunction. One important disadvantage of this correlation is that it is not translationally invariant unless the radial degrees of freedom is frozen. The Hamiltonian model has solutions which have this correlation built in. Intuitively the correlation can be understood easily by imagining objects with associated 'arrows'. The arrows cannot be oriented either parallel or anti-parallel to each other. The model has some interesting features and it would be of great interest to find physical systems which incorporate these features.

In this paper we elaborate on our earlier results [13] and present several new results. In section 2, we discuss the many-body Hamiltonian and display some of the exact solutions and their structure. The similarities between the spectrum of these exact solutions and the spectrum of CSM are quite remarkable. Further, when projected onto a circle the model reduces to a variant of the trigonometric Sutherland model. In this limit the model also has translational invariance. In section 3, we discuss the two-body problem in detail and show that the solutions of the two-body problem are described by the Heun equation. In particular, the spectrum becomes very simple for large values of the interaction strength. The singular interaction discussed in this paper requires careful treatment in the region near $X_{ij} = 0$; this is discussed in the appendix [14]. Section 4 contains a discussion and summary.

2. The many-body Hamiltonian and some exact solutions

For the sake of completeness we recall first the Hamiltonian and some of its properties that have been proposed [13]. We also clarify some points which were not made explicitly clear in the earlier paper. The N-particle Hamiltonian which we consider is given by

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N}\boldsymbol{r}_i^2 + \frac{\hbar^2}{2m}g_1\sum_{\substack{i,j\\(i\neq j)}}^{N}\frac{\boldsymbol{r}_j^2}{X_{ij}^2} + \frac{\hbar^2}{2m}g_2\sum_{\substack{i,j,k\\(i\neq j\neq k)}}^{N}\frac{\boldsymbol{r}_j\cdot\boldsymbol{r}_k}{X_{ij}X_{ik}} \tag{2}$$

where X_{ij} is given by (1); g_1 and g_2 are dimensionless coupling strengths of the two- and three-body interactions respectively. While g_1 and g_2 can be independent of each other in

general, for the type of solutions involving the correlator in (1) they are not. We will specify their relationship shortly. The particles are confined in a one-body oscillator confinement potential. The Hamiltonian is rotationally invariant and manifestly symmetric in all particle indices. As in the CSM, we may scale away the mass, m, and oscillator frequency, ω, by scaling all distances $\boldsymbol{r}_i \to \sqrt{m\omega/\hbar}\boldsymbol{r}_i$, and measuring the energy in units of $\hbar\omega$. This is done by setting $\hbar = 1$, $m = 1$, and $\omega = 1$. In these units, the Hamiltonian is given by

$$H = -\frac{1}{2}\sum_{i=1}^{N}\boldsymbol{\nabla}_i^2 + \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{r}_i^2 + \frac{g_1}{2}\sum_{\substack{i,j\\(i\neq j)}}^{N}\frac{\boldsymbol{r}_j^2}{X_{ij}^2} + \frac{g_2}{2}\sum_{\substack{i,j,k\\(i\neq j\neq k)}}^{N}\frac{\boldsymbol{r}_j\cdot\boldsymbol{r}_k}{X_{ij}X_{ik}}\ . \tag{3}$$

Note that the total angular momentum operator $L = \sum_i (x_i p_{y_i} - y_i p_{x_i})$ commutes with the Hamiltonian since it is rotationally invariant, and may therefore be used to label the states. The Hamiltonian is invariant under parity $x \to -x$ and $y \to y$. In addition, for any i, the Hamiltonian is invariant under the transformation $\boldsymbol{r}_i \to -\boldsymbol{r}_i$ and $\boldsymbol{r}_k \to \boldsymbol{r}_k$ for all $k \neq i$. This D_{2N} invariance is special to this system, and we are not aware of any other interacting many-body Hamiltonian which has this symmetry. The consequences of this will be discussed explicitly in the two-body problem where this is related to the supersymmetric properties of the system.

We will consider both bosonic and fermionic systems governed by the Hamiltonian (3), i.e. wavefunctions which are totally symmetric and asymmetric respectively. It will turn out that certain calculations (for example, in the two-body problem) simplify if we do not impose any symmetry to begin with.

2.1. The exact bosonic ground state

We first obtain the exact bosonic ground state of this Hamiltonian. As an ansatz for the ground-state wavefunction, consider a solution of the form

$$\Psi_0(x_i, y_i) = \prod_{i<j}^{N}|X_{ij}|^g \exp\left(-\tfrac{1}{2}\sum_{i=1}^{N}\boldsymbol{r}_i^2\right). \tag{4}$$

Clearly Ψ_0 correctly incorporates the behaviour of the wavefunction in the asymptotic region $|\boldsymbol{r}_i| \to \infty$, and Ψ_0 is regular for $g \geqslant 0$. In general we insist that our solutions have this asymptotic form; the conditions under which this is valid will be specified later. The eigenvalue equation now takes the form

$$H\Psi_0 = \left[\tfrac{1}{2}(g_1 - g(g-1))\sum_{\substack{i,j\\(i\neq j)}}^{N}\frac{\boldsymbol{r}_j^2}{X_{ij}^2} + \tfrac{1}{2}(g_2 - g^2)\sum_{\substack{i,j,k\\(i\neq j\neq k)}}^{N}\frac{\boldsymbol{r}_j\cdot\boldsymbol{r}_k}{X_{ij}X_{ik}} + gN(N-1) + N\right]\Psi_0. \tag{5}$$

Therefore Ψ_0 is the exact many-body ground state for an arbitrary number of particles of the Hamiltonian if

$$g_1 = g(g-1) \qquad \text{and} \qquad g_2 = g^2. \tag{6}$$

Since $g \geqslant 0$, we have $g_1 \geqslant -\frac{1}{4}$ and g_2 is positive definite. Note that the range of g_1 is identical to the one obtained in the CSM. The ground-state energy is now given by

$$E_0 = N + gN(N-1). \tag{7}$$

Note that this has exactly the form of the ground-state energy of the CSM.

Since g determines both g_1 and g_2 uniquely, we will regard g as the fundamental parameter of the Hamiltonian which determines the strength of the interaction. In other words, we demand that the ground state should be given by (4), and we *define* the Hamiltonian to ensure that. It turns out that such a definition requires some special care in the vicinity of $X_{ij} = 0$ (called the ultraviolet region below). In the appendix we will discuss this for the case of two particles. Note that in the two-particle case, the Hamiltonian only contains the parameter $g_1 = g(g-1)$ and not g_2. As a result, for every value of g_1 in the range $-\frac{1}{4} < g_1 < 0$, the bosonic ground-state energy, given by $E_0 = 2 + 2g$, has two possible values; these two possibilities correspond to different potentials in the ultraviolet region. This is somewhat unusual but it is not uncommon for singular potentials. The same thing also happens in the CSM even for the N-body problem; see for example [15–17]. We discuss this issue in detail in the appendix where we show that the ultraviolet regularization is determined by the parameter g rather than by g_1.

We emphasize that our objective here is not to find the general solutions for arbitrary g_1 and g_2, but to find a Hamiltonian whose solutions have the novel correlation in equation (1) built in. In general, if g_1 and g_2 are independent, the Hamiltonian will have a ground state different from the one given above. Our procedure is therefore similar to the many-anyon problem where also there are two- and three-body interactions, but the strengths are related to a single parameter. With the form of g_1 and g_2 given in (6), the solution found above is indeed the lowest energy state.

A neat way of proving that we have indeed obtained the ground state can be given using the method of operators [18]. To this end, define the operators

$$\begin{aligned} Q_{x_i} &= p_{x_i} - \mathrm{i}x_i + \mathrm{i}g \sum_{j(j\neq i)} \frac{y_j}{X_{ij}} \\ Q_{y_i} &= p_{y_i} - \mathrm{i}y_i - \mathrm{i}g \sum_{j(j\neq i)} \frac{x_j}{X_{ij}} \end{aligned} \tag{8}$$

and their Hermitian conjugates $Q^\dagger_{x_i}$ and $Q^\dagger_{y_i}$. It is easy to see that the Q's annihilate the ground state in equation (4), $Q_{x_i}\Psi_0 = 0$ and $Q_{y_i}\Psi_0 = 0$. The Hamiltonian can now be recast in terms of these operators as

$$\tfrac{1}{2}\sum_i [Q^\dagger_{x_i} Q_{x_i} + Q^\dagger_{y_i} Q_{y_i}] = H - E_0 \tag{9}$$

where E_0 is given by equation (7). Clearly the operator on the left-hand side is positive definite and annihilates the ground-state wavefunction given by equation (4). Therefore E_0 must be the minimum energy that an eigenstate can have.

As we remarked earlier, the ground state of the Hamiltonian is bosonic. The ground state of the Hamiltonian for a fermionic system is not easy to determine analytically (for $g > 0$). The problem here is analogous to a similar problem in the many-anyon Hamiltonian [19–21]. In section 3, we will determine the fermionic ground-state energy for two particles both numerically and to first order in g using the perturbation theory near $g = 0$ and show that it has quite an unusual behaviour.

2.2. Spectrum of excited states

While we have not been able to find the complete excited-state spectrum of the model, the eigenvalue equation for a general excited state may be obtained as follows. From the asymptotic properties of the solutions of the Hamiltonian in equation (3), it is clear that Ψ has the general structure $\Psi(x_i, y_i) = \Psi_0(x_i, y_i)\Phi(x_i, y_i)$, where Ψ_0 is the ground-state

wavefunction. Obviously if Φ is a constant we recover the ground state. In general, Φ satisfies the eigenvalue equation

$$\left[-\frac{1}{2}\sum_{i=1}^{N}\boldsymbol{\nabla}_i^2 + \sum_{i=1}^{N}\boldsymbol{r}_i\cdot\boldsymbol{\nabla}_i + g\sum_{\substack{i,j\\(i\neq j)}}\frac{1}{X_{ij}}\left(x_j\frac{\partial}{\partial y_i} - y_j\frac{\partial}{\partial x_i}\right)\right]\Phi = (E-E_0)\Phi. \tag{10}$$

It is interesting to note that while g_1 is zero both at $g = 0$ and 1, the term containing g in the above expression is zero only when $g = 0$. This is so because of the boundary condition that the wavefunctions must vanish as $|X_{ij}|^g$ for nonzero g.

We first discuss the exact solutions of the above differential equation. This is easily done by defining the complex coordinates $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$, and their partial derivatives $\partial = \partial/\partial z = \frac{1}{2}(\partial/\partial x - \mathrm{i}\partial/\partial y)$, and $\partial^* = \partial/\partial z^* = \frac{1}{2}(\partial/\partial x + \mathrm{i}\partial/\partial y)$. In these coordinates, the differential equation for Φ reduces to $\tilde{H}\Phi = (E - E_0)\Phi$, where

$$\tilde{H} = -2\sum_i \partial_i\partial_i^* + \sum_i (z_i\partial_i + z_i^*\partial_i^*) + 2g\sum_{\substack{i,j\\(i\neq j)}}\frac{z_j\partial_i - z_j^*\partial_i^*}{z_i z_j^* - z_j z_i^*}. \tag{11}$$

In addition, Φ is an eigenstate of the total angular momentum operator, $L\Phi = l\Phi$. We can now classify some exact solutions according to their angular momentum.

(a) $l = 0$ solutions: Define an auxiliary parameter $t = \sum_i z_i z_i^*$, and let $\Phi = \Phi(t)$. This has zero total angular momentum. The differential equation for Φ reduces to

$$t\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + (b-t)\frac{\mathrm{d}\Phi}{\mathrm{d}t} - a\Phi = 0 \tag{12}$$

where $b = E_0$ and $a = (E_0 - E)/2$; E_0 is the energy of the ground state. The allowed solutions are the regular confluent hypergeometric functions [22], $\Phi(t) = M(a, b, t)$. Normalizability imposes the restriction $a = -n_r$, where n_r is a positive integer; then $\Phi(t)$ is a polynomial of degree n_r (the subscript 'r' denotes radial excitations as discussed later). The corresponding eigenvalues are $E = E_0 + 2n_r$. This class of solutions was discussed in [13].

(b) $l > 0$ solutions: Let $t_z = \sum_i z_i^2$, and let $\Phi = \Phi(t_z)$. The total angular momentum is not zero. All the mixed derivative terms in equation (11) drop out, and we obtain the differential equation

$$2t_z\frac{\mathrm{d}\Phi}{\mathrm{d}t_z} = (E - E_0)\Phi. \tag{13}$$

This is the well known Euler equation whose solutions are just monomials in t_z. The solution is given by $\Phi(t_z) = t_z^m$, and the total angular momentum is $l = 2m$. The eigenvalues are $E = E_0 + 2m = E_0 + l$.

(c) $l < 0$ exact solutions: Let $t_{z^*} = \sum_i (z_i^*)^2$, and let $\Phi = \Phi(t_{z^*})$. Once again the differential equation for Φ reduces to

$$2t_{z^*}\frac{\mathrm{d}\Phi}{\mathrm{d}t_{z^*}} = (E - E_0)\Phi. \tag{14}$$

This is similar to the previous case. The solution is given by $\Phi(t_{z^*}) = t_{z^*}^m$, and the total angular momentum is $l = -2m$. The eigenvalues are $E = E_0 + 2m = E_0 - l$.

(d) Tower of excited states: One can now combine solutions of a given l in cases (b) or (c) with the solutions in (a), and obtain a new class of excited states. Let us define $\Phi(z_i, z_i^*) = \Phi_1(t)\Phi_2(t_z)$, where Φ_1 is the solution with $l = 0$, Φ_2 is the solution with $l > 0$,

and t and t_z have been defined already. The differential equation for Φ is again a confluent hypergeometric equation given by

$$t\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + (b-t)\frac{\mathrm{d}\Phi}{\mathrm{d}t} - a\Phi = 0 \tag{15}$$

where $b = E_0 + 2m$ and $a = (E_0 + 2m - E)/2$. The energy eigenvalues are then given by $E = E_0 + 2n_r + 2m = E_0 + 2n_r + l$. One may repeat the procedure to obtain exact solutions for a tower of excited states with $l < 0$ solutions. As we shall see below, the existence of the tower is a general result applicable to all excited states of which the exact solutions shown above form a subset. We notice that these solutions bear a remarkable resemblance to the many-anyon system where a similar structure exists for the known class of exact solutions [12].

(e) A general class of excited states: One can combine the solutions of all the three classes (a), (b) and (c) to obtain an even more general class of solutions. Consider the polynomial $P(n_1, n_2, n_3) = t^{n_1} t_z^{n_2} t_{z^*}^{n_3}$, where the n_i are nonnegative integers. Using the form of (11), one can show that

$$\begin{aligned}\tilde{H}P(n_1, n_2, n_3) &= 2(n_1 + n_2 + n_3)P(n_1, n_2, n_3) - 8n_2n_3P(n_1 + 1, n_2 - 1, n_3 - 1) \\ &\quad -2n_1[n_1 + 2n_2 + 2n_3 + gN(N-1)]P(n_1 - 1, n_2, n_3).\end{aligned} \tag{16}$$

Using this one can show that there is an exact polynomial solution, whose highest degree term is $P(n_1, n_2, n_3)$. The energy of this solution is $E = E_0 + 2(n_1 + n_2 + n_3)$, and the angular momentum is $l = 2(n_2 - n_3)$.

While there may be more exact solutions, we do not know of a simple way of solving them. We can however gather some general features as follows. The coordinates (x_i, y_i) can be separated into one 'radial' coordinate $t = \sum_i r_i^2$ as above and $2N - 1$ 'angular' coordinates collectively denoted by Ω_i(say). Then, equation (10) can be expressed as

$$t\frac{\partial^2\Phi}{\partial t^2} + (E_0 - t)\frac{\partial\Phi}{\partial t} - \frac{1}{t}\,\mathcal{L}\Phi + \frac{1}{2}(E - E_0)\Phi = 0 \tag{17}$$

where $\mathcal{L} = \mathcal{D}_2 + g\mathcal{D}_1$, and $\mathcal{D}_n$ is an nth-order differential operator which only acts on functions of the angles Ω_i. In particular, $\mathcal{D}_2$ is the Laplacian on a sphere of dimension $2N - 1$. Next we note that Φ can be factorized in the form $\Phi(x_i, y_i) = R(t)Y(\Omega_i)$, where Y, a generalized spherical harmonic defined on the $(2N-1)$-dimensional sphere S^{2N-1}, satisfies the eigenvalue equation $\mathcal{L}Y = \lambda Y$. (This is the hard part of the spectral problem, to find the eigenvalues λ.) We now define

$$\mu = \sqrt{(E_0 - 1)^2 + 4g\lambda} - (E_0 - 1). \tag{18}$$

Further if we write $R(t) = t^{\mu/2}\tilde{R}(t)$, then $\tilde{R}$ satisfies a confluent hypergeometric equation

$$t\frac{\mathrm{d}^2\tilde{R}}{\mathrm{d}t^2} + (b-t)\frac{\mathrm{d}\tilde{R}}{\mathrm{d}t} - a\tilde{R} = 0 \tag{19}$$

where $b = E_0 + \mu$ and $a = (E_0 + \mu - E)/2$. The admissible solutions are the regular confluent hypergeometric functions, $\tilde{R}(t) = M(a, b, t)$. Normalizability imposes the restriction $a = -n_r$, where n_r is a positive integer. Then $\tilde{R}(t)$ is a polynomial of degree n_r, and it has n_r nodes. The energy of this state is given by $E = E_0 + \mu + 2n_r$. We see that for a given value of μ, there is an infinite tower of energy eigenvalues separated by a spacing of 2. As remarked earlier, this is reminiscent of what happens in the case of anyons. The tower structure and the angular momentum are useful in organizing a numerical or analytical study of the energy spectrum. Since the radial quantum number, n_r, and the angular momentum, l, are integers, they cannot change as the parameter, g, is varied continuously.

2.3. Relation to the Sutherland model

It may be of interest to note that the model reduces to a variant of the Sutherland model [2] in one dimension. In this limit, therefore the model is exactly solvable. Restricting the particles to move along the perimeter of a unit circle in the Hamiltonian (3) without the confinement potential, we obtain

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial\theta_i^2} + \frac{g_1}{2}\sum_{\substack{i,j\\(i\neq j)}}^{N}\frac{1}{\sin^2(\theta_i-\theta_j)} + \frac{g_2}{2}\sum_{\substack{i,j,k\\(i\neq j\neq k)}}^{N}[1+\cot(\theta_i-\theta_j)\cot(\theta_i-\theta_k)] \tag{20}$$

since $X_{ij} = -\sin(\theta_i-\theta_j)$ now. By using the identity

$$\sum_{\substack{i,j,k\\(i\neq j\neq k)}}^{N}\cot(\theta_i-\theta_j)\cot(\theta_i-\theta_k) = -\frac{N(N-1)(N-2)}{3} \tag{21}$$

we immediately recover an analogue of the trigonometric Sutherland model, but shifted by the constant $g_2\,N(N-1)(N-2)/3$. Note, however, that the potential in (20) depends on the function $\sin(\theta_i-\theta_j)$, rather than the chord length which is proportional to $\sin[(\theta_i-\theta_j)/2]$. Interestingly, the wavefunction has twice the periodicity of the Sutherland model solutions—the wavefunction vanishes whenever the particles are at diametrically opposite points on a circle or at the same point.

3. The two-body problem: complete solution

While we have not been able to solve the many-body problem completely, the two-body problem in our model is exactly solvable. We demonstrate this by going over to the hyperspherical formalism first proposed in two dimensions by Kilpatrick and Larsen [23] (see also [19]). We discuss some of the properties of the two-body spectrum. We also explicitly show that the two-body problem is integrable. It is important to note that the two-particle interaction is sufficiently singular and that careful treatment is required in order to define the problem completely consistently; this is described in the appendix.

The two-body Hamiltonian is given by

$$H = -\frac{1}{2}[\boldsymbol{\nabla}_1^2+\boldsymbol{\nabla}_2^2] + \frac{1}{2}[\boldsymbol{r}_1^2+\boldsymbol{r}_2^2] + \frac{g_1}{2}\frac{\boldsymbol{r}_1^2+\boldsymbol{r}_2^2}{X^2} \tag{22}$$

where $X = x_1y_2 - x_2y_1$. The two-body problem is solved best in the hyperspherical coordinate system which allows a parametrization of the coordinates $\boldsymbol{r}_1, \boldsymbol{r}_2$ in terms of three angles and one length, (R,θ,ϕ,ψ) as follows:

$$\begin{aligned} x_1 + \mathrm{i}y_1 &= R(\cos\theta\cos\phi - \mathrm{i}\sin\theta\sin\phi)\exp(\mathrm{i}\psi)\\ x_2 + \mathrm{i}y_2 &= R(\cos\theta\sin(\phi) + \mathrm{i}\sin\theta\cos\phi)\exp(\mathrm{i}\psi). \end{aligned} \tag{23}$$

We may regard (R,θ,ϕ) as the body-fixed coordinates which are transformed to the space-fixed system by an overall rotation of ψ. For a fixed R, these coordinates define a sphere in four dimensions within the following intervals:

$$-\pi/4 \leqslant \theta \leqslant \pi/4 \qquad -\pi/2 \leqslant \phi \leqslant \pi/2 \qquad \text{and} \qquad -\pi \leqslant \psi \leqslant \pi. \tag{24}$$

An exchange of two particles is achieved by

$$\begin{aligned} &\theta \to -\theta \\ &\phi \to \pi/2 - \phi \qquad \psi \to \psi \qquad \text{if } \phi > 0 \\ &\phi \to -\pi/2 - \phi \qquad \psi \to \pi + \psi \qquad \text{if } \phi < 0. \end{aligned} \tag{25}$$

With this choice of coordinates, the radial coordinate becomes $R^2 = r_1^2 + r_2^2$, which is the radius of the sphere in four dimensions. Also, $X = x_1 y_2 - x_2 y_1 = R^2 \sin(2\theta)/2$. Notice that X depends only on R and θ. Therefore the two-body interaction in the Hamiltonian is independent of the angles ϕ and ψ. The integrals of the motion of the system may be constructed in terms of these new coordinates. The angular momentum operator is given by $L = \sum_i (x_i p_{y_i} - y_i p_{x_i}) = -\mathrm{i}\partial/\partial\psi$, which commutes with the Hamiltonian. Another constant of motion exists and is given by

$$Q = \mathrm{i}\left[x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2}\right] = -\mathrm{i}\frac{\partial}{\partial \phi}. \tag{26}$$

Since Q is asymmetric, acting on a symmetric state produces an asymmetric state and vice versa. We therefore refer to this as a supersymmetry operator (SUSY). The operator Q is similar to the SUSY operator discovered in the many-anyon problem by Sen [24]. Note that the differential operator for both angular momentum and the SUSY operators has a very simple form in the hyperspherical coordinates. The states can therefore be labelled by the quantum numbers associated with these two operators which we denote by l and q respectively. With SUSY, the two-body problem is integrable. (The four constants of motion are the Hamiltonian, H, the angular part of H, L and Q.) Note that we have $QX = 0$ which makes calculations simple. It is easy to check that the bosonic ground state of the Hamiltonian has the quantum numbers l and q of the angular momentum and SUSY operators equal to zero.

We would like to emphasize that the eigenstates of the SUSY operator Q are neither symmetric nor asymmetric, unless the eigenvalue $q = 0$. After finding a simultaneous eigenstate of H, L and Q, we can separate it into symmetric (bosonic) and asymmetric (fermionic) parts. These parts are individual eigenstates of Q^2 but not of Q. Specifically, we have $Q\Psi_B = q\Psi_F$ and $Q\Psi_F = q\Psi_B$, where B and F denote bosonic and fermionic states respectively. Then $\Psi_B \pm \Psi_F$ are eigenstates of Q, while Ψ_B and Ψ_F are eigenstates of Q^2.

The two-body Hamiltonian in terms of the hyperspherical coordinates is given by

$$H = -\frac{1}{2}\left[\frac{\partial^2}{\partial R^2} + \frac{3}{R}\frac{\partial}{\partial R} - \frac{\Lambda^2}{R^2} - R^2\right] + g_1 \frac{2}{R^2 \sin^2(2\theta)} \tag{27}$$

where the operator Λ^2 is the Laplacian on the sphere S^3 and is given by

$$-\Lambda^2 = \frac{\partial^2}{\partial\theta^2} - \frac{2\sin(2\theta)}{\cos(2\theta)}\frac{\partial}{\partial\theta} + \frac{1}{\cos^2(2\theta)}\left[\frac{\partial^2}{\partial\phi^2} + 2\sin(2\theta)\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right]. \tag{28}$$

The interaction in the Hamiltonian is independent of the angles ϕ, ψ and depends only on R, θ. The operators L and Q commute with the Hamiltonian since they commute with the noninteracting ($g = 0$) Hamiltonian. We thus label the states with the eigenvalues of these operators for all g_1. Each of these states is four-fold degenerate: Under parity, $L \to -L$ and $Q \to Q$ and the Hamiltonian is invariant under parity. Therefore the states labelled by quantum numbers (l, q) have the same energy as $(-l, q)$. The Hamiltonian is also invariant under the transformation $\boldsymbol{r}_1 \to -\boldsymbol{r}_1$ and $\boldsymbol{r}_2 \to \boldsymbol{r}_2$. This is a discrete symmetry particular to this system. Under this transformation $L \to L$ and $Q \to -Q$. Therefore the

states labelled by quantum numbers (l, q) have the same energy as $(l, -q)$. Combining the two we obtain the four-fold degeneracy of the states. Later we will find that the states with (l, q) have the same energy as (q, l) since interchanging q and l leaves the differential equation invariant; therefore the energy of these two states must be the same. We thus have an eight-fold degeneracy for the levels for which $|q|$ and $|l|$ are nonzero and different from each other. Note that this degeneracy is a subset of the degeneracy of the noninteracting system. If $|l| = |q|$ is nonzero, we have a four-fold degeneracy. Finally, there is a four-fold degeneracy between the states $(\pm l, 0)$ and $(0, \pm l)$ if $l \neq 0$.

3.1. Solutions of the eigenvalue equation

We are now interested in solving the eigenvalue equation given by $H\Psi = E\Psi$. Following the remarks made in the previous section, we may, in general, write $\Psi = F(R)\Phi(\theta, \phi, \psi)$. The eigenvalue equation separates into angular and radial equations. The angular equation is given by

$$\left(\Lambda^2 + \frac{4g_1}{\sin^2(2\theta)}\right)\Phi = \beta(\beta+2)\Phi \tag{29}$$

where $\beta \geqslant -1$, and the radial equation is given by

$$\frac{\mathrm{d}^2F}{\mathrm{d}R^2} + \frac{3}{R}\frac{\mathrm{d}F}{\mathrm{d}R} + \left(2E - R^2 - \frac{\beta(\beta+2)}{R^2}\right)F = 0. \tag{30}$$

The radial equation can be easily solved by using the methods outlined in the last section of [22]. The solution is given by

$$F(R) = R^\beta M(a, b, R^2)\exp(-R^2/2) \tag{31}$$

where $b = \beta + 2$ and $a = (\beta + 2 - E)/2$ and $M(a, b, R^2)$ is the confluent hypergeometric function. Demanding that $a = -n_r$ where n_r is an integer, the energy is given by

$$E = 2n_r + \beta + 2. \tag{32}$$

Note that β is still unknown and has to be obtained by solving the angular equation. Nevertheless, the tower structure of the eigenvalues built on radial excitation of the ground states is obvious from the above.

The angular equation (29) may be solved with the ansatz

$$\Phi(\theta, \phi, \psi) = P(x)\exp(\mathrm{i}q\phi)\exp(\mathrm{i}l\psi) \tag{33}$$

where $x = \sin(2\theta)$ and l, q are the state labels in terms of the integer valued eigenvalues of the angular momentum and SUSY operators. The angular equation then reduces to a differential equation in a single variable, x, for the function $P(x)$:

$$(1-x^2)\frac{\mathrm{d}^2P}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P}{\mathrm{d}x} - \frac{1}{4(1-x^2)}[q^2 + 2xql + l^2]P - \frac{g_1}{x^2}P + \frac{\beta(\beta+2)}{4}P = 0. \tag{34}$$

Note that the equation has four regular singularities at $x = 0, 1, -1, \infty$ (the singularity at ∞ does not play any role since x is bounded). Therefore the solution is of the form

$$P(x) = |x|^a(1-x)^b(1+x)^c\Theta^{a,b,c}(x). \tag{35}$$

One can now fix a, b, c to cancel the singularities. We find that $b = |l + q|/4$ and $c = |l - q|/4$. Since l and q are integer valued, the values of b and c are restricted. The other exponent a is given by

$$a(a-1) = g_1 = g(g-1) \qquad \text{with } a \geqslant 0 \tag{36}$$

where we have already defined g_1 through equation (6) in terms of g. Note that we have used the symbol a instead of g. As shown in the appendix, we have to take $a = g$ if $g \geqslant \frac{1}{2}$. But if $g < \frac{1}{2}$, we have to generally consider a linear superposition of solutions with a equal to g and $1 - g$ (there are more details on this later).

We finally arrive at the required differential equation from which the eigenvalues are determined,

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} + 2[a/x - (b-c) - (a+b+c+1)x]\frac{\mathrm{d}\Theta}{\mathrm{d}x} + \left[\frac{(\beta+1)^2}{4} - (a+b+c+\tfrac{1}{2})^2 + 2a(c-b)/x\right]\Theta = 0. \tag{37}$$

For $g = 0$, the solutions are simply Jacobi polynomials and the full solution for the angular part is given in terms of the spherical harmonics on a four-dimensional sphere. In general, this differential equation is known as the Heun equation whose solutions $\Theta^{a,b,c}(x)$ are characterized by the so-called P-symbols [25]. The Heun equation is exactly solvable if either l or q vanishes, i.e. if $b = c$ as discussed in the next section. The equation is also exactly solvable at an infinite number of isolated points in the space of parameters (a, b, c). These are isolated points because if we vary a slightly away from any one of them, the equation is not exactly solvable. Note that b, c take discrete values and cannot be varied continuously.

3.2. Polynomial solutions

Let us first consider a class of solutions which are polynomials in x. We may then write

$$\Theta(x) = \sum_{k=0}^{p} C_k x^k \tag{38}$$

where we may define $C_0 = 1$. Substituting this into the differential equation for Θ, we see that the $C_k's$ satisfy a three-term recursion relation given by

$$(k+2)(k+1+2a)\ C_{k+2} - 2(b-c)(k+1+a)C_{k+1} + \left[\frac{(\beta+1)^2}{4} - (a+b+c+k+\tfrac{1}{2})^2\right]C_k = 0 \tag{39}$$

which is, in general, difficult to solve. However, there are two special cases when polynomial solutions are possible. (i) For $b = c$, this reduces to a two-term recursion relation which can be easily solved to obtain all the energy levels. This is an example of a conditionally exactly solvable (CES) problem [26, 27] in which the full spectrum is exactly solvable for some special condition (such as $b = c$ here). (ii) The other case is when the coefficient of C_k is zero with $k = p$ (where $p \geqslant 1$), i.e.

$$E = 2n_r + 2a + 2b + 2c + 2p + 2 \tag{40}$$

in which case one has a polynomial solution of degree p. This is an example of a quasi-exactly solvable (QES) problem when only a finite number of states are exactly solvable for some given values of the parameters. As far as we are aware, this is the first example where both CES and QES solutions exist in the same problem. We now discuss both types of solutions in detail.

(i) CES-type solutions: To see the solutions explicitly, define $y = x^2$. In terms of the variable y the differential equation is written as,

$$y(1-y)\frac{\mathrm{d}^2\Theta}{\mathrm{d}y^2} + 2[a' - (b'+c'+1)y]\frac{\mathrm{d}\Theta}{\mathrm{d}y} - b'c'\Theta = 0 \tag{41}$$

where

$$\begin{aligned} a' &= a + \tfrac{1}{2} \\ b' &= \frac{1}{2}\left[a + 2b + 1 + \frac{\beta}{2}\right] \\ c' &= \frac{1}{2}\left[a + 2b - \frac{\beta}{2}\right]. \end{aligned} \tag{42}$$

This is now a hypergeometric equation whose solutions are given by $F(b', c', a'; y)$. Note that we need to concentrate only on the solutions for $0 \leqslant y \leqslant 1$. The hypergeometric series terminates whenever b' or c' is a negative integer or zero. In our case b' is always positive hence the solutions are given when $c' = -m$, where m is an integer. Therefore $\beta = 2m + 2a + 4b$. The energy eigenvalues are obtained by substituting this into equation (32),

$$E = 2n_r + 2a + 4b + 2m + 2. \tag{43}$$

Since $b = c$, we must have either $l = 0$ or $q = 0$. The energy varies linearly with a as in the case of the exact solutions of the many-body problem. The expression for a in terms of g will be clarified in the next section. It might be tempting to conclude that all polynomial solutions vary linearly with g. In fact, that is not so as can be seen from the following examples.

(ii) QES-type solutions: We can have polynomial solutions of degree p (where $p \geqslant 1$) if b, c and g satisfy some specific relations. Let us consider the case $\Theta(x) = 1 + \delta x$. This is a solution if

$$a = \frac{b + c}{(b - c)^2 - 1} - 1 \tag{44}$$

and $\delta = b - c$. We have implicitly assumed that b is not equal to c since otherwise this solution is trivial and of CES type. (Once again, a can be equal to either g or $1 - g$ as discussed in the previous section.) Then $\beta = 2a + 2b + 2c + 2$, and the full spectrum is given by $E = 2n_r + 2a + 2b + 2c + 4$. We should point out that a solution of this kind is only possible for fairly large values of l and q; the minimum values needed are $|l| = |q| = 3$, in which case $a = \frac{1}{5}$.

Similarly, there is a polynomial solution of the form $\Theta(x) = 1 + \delta x + \epsilon x^2$, if

$$a = 2\alpha - \tfrac{3}{2} + \sqrt{4\alpha^2 - \alpha + \tfrac{1}{4}} \qquad \text{where } \alpha = \frac{b + c}{(b - c)^2 - 4}. \tag{45}$$

The spectrum is given by $E = 2n_r + 2a + 2b + 2c + 6$. These expressions for the energies are nonlinear in a because of constraint (44) or (45). We should however caution that these are isolated solutions since b and c can only take discrete values; hence the above solutions do not vary smoothly with a. In general, solutions similar to the above may be constructed for every degree p of the polynomial; the corresponding energies are given by equation (40) where a is given by a function of b and c which can be derived by solving $p + 1$ recursion relations obtained from equations (39) by setting $k = -1, 0, 1, \ldots, p - 1$.

3.3. Numerical analysis

We now consider the numerical solution of some low-lying states of the two-body problem since the polynomial solutions described in the previous section do not exhaust the full spectrum.

The noninteracting limit of the system is $g = 0$ where we have the solutions corresponding to a four-dimensional oscillator. These are simply the spherical harmonics on a four-sphere $Y_{k,l,q}$, where $k = 0, 1, 2, \cdots$ and $|l|, |q| \leqslant k$ label the states. When the degeneracy of these states is taken into account, all the states in the noninteracting limit $g = 0$ are completely specified. We now demand that the wavefunctions and energy levels should vary continuously with the parameter g which is the interaction strength. We also require that the wavefunctions should not diverge at any value of x in the interval $[-1, 1]$.

For given values of $b \neq c$, we can numerically find the energy levels in two different ways. We can diagonalize the differential operator in (34) in the basis of the noninteracting ($g = 0$) states, or we can solve the differential equation (34) or (37) directly for each state. We have used both methods and will present the results below.

In order to proceed further, it is necessary to clarify the dependence of a on g in equation (36). By the arguments given in the appendix, $a = g$ if $g \geqslant \frac{1}{2}$. If $g < \frac{1}{2}$, a can be equal to either g or $1 - g$; for any given state, we choose the value which is continuously connected to the noninteracting solution at $g = 0$. Namely, any solution of the above kind must, at $g = 0$, go either as 1 or as x near $x = 0$; we then choose $a = g$ or $a = 1 - g$ in the two cases respectively for $0 < g < \frac{1}{2}$. In particular, an exact solution which, near $x = 0$, goes as 1 at $g = 0$ will go as $|x|^g$ for all $g > 0$. An exact solution which goes as x at $g = 0$ will go as $|x|^{1-g}$ upto $g = \frac{1}{2}$ and then as $|x|^g$ for $g > \frac{1}{2}$. As a result solutions of this form are discontinuous at $g = \frac{1}{2}$. In general, when solving numerically for the nonexact solutions, we have to allow a superposition of both $|x|^g$ and $|x|^{1-g}$. For $g \geqslant \frac{1}{2}$, however, a must be equal to g.

When solving differential equation (34) or (37), we have to consider the two regions separately. According to the rules discussed in the appendix, for $0 \leqslant g < \frac{1}{2}$, we take the function $P(x)$ to go as

$$P(x) = |x|^g + d\,\mathrm{sgn}(x)|x|^{1-g} \tag{46}$$

near $x = 0$, and we vary both the coefficient d as well as β in equation (34) until we find that the solution of (34) does not diverge at $x = \pm 1$. Both d and β depend on g; it is possible to determine the limiting values $d(0)$ and $d(\frac{1}{2})$ as follows. At $g = 0$, suppose that the Jacobi polynomial is normalized so that $P = 1$ and $P' = C_J$ at $x = 0$. On the other hand, from equation (39), we see that $C_1 = b - c$ for any nonzero g, no matter how small. By taking the limit $g \to 0$ in equation (46), we therefore find that $d(0) = C_J - b + c$. At the other end, $d(\frac{1}{2}) = \pm 1$ because, as we will show next, $P(x)$ must vanish for either $x \geqslant 0$ or for $x \leqslant 0$ for all $g \geqslant \frac{1}{2}$ (except for the exact polynomial solutions discussed previously).

For $g \geqslant \frac{1}{2}$, we must take the solution of the differential equation to go either as (a) $P(x) = 0$ for $x \leqslant 0$ and $\sim x^g$ for x small and positive, or as (b) $P(x) = 0$ for $x \geqslant 0$ and $\sim (-x)^g$ for x small and negative. In either case, we vary the energy until we find that the solution of (37) (with $a = g$) does not diverge at $x = 1$ and -1 for (a) and (b) respectively. It is interesting to note that for each such solution, $P(x)$ vanishes identically in one of the half intervals $[-1, 0]$ or $[0, 1]$. Note also that if $\Theta^{a,b,c}(x)$ is a solution which is only nonzero for $x \geqslant 0$, then $\Theta^{a,c,b}(x) = \Theta^{a,b,c}(-x)$ will be a solution which is only nonzero for $x \leqslant 0$; further, the two solutions will have the same energy.

In contrast to the above method for solving the Heun equation (37) directly, the numerical diagonalization procedure for finding the eigenvalues involves solving the eigenvalue equation (29). The basis for diagonalization is provided by the eigenstates of the Laplacian Λ^2 on S^3, namely the spherical harmonics on a four-sphere $Y_{k,l,q}$. For nonzero interaction strengths, the singular interaction is handled by multiplying the noninteracting eigenstates by $|x|^g$. The resulting basis is nonorthogonal, and the diagonalization procedure

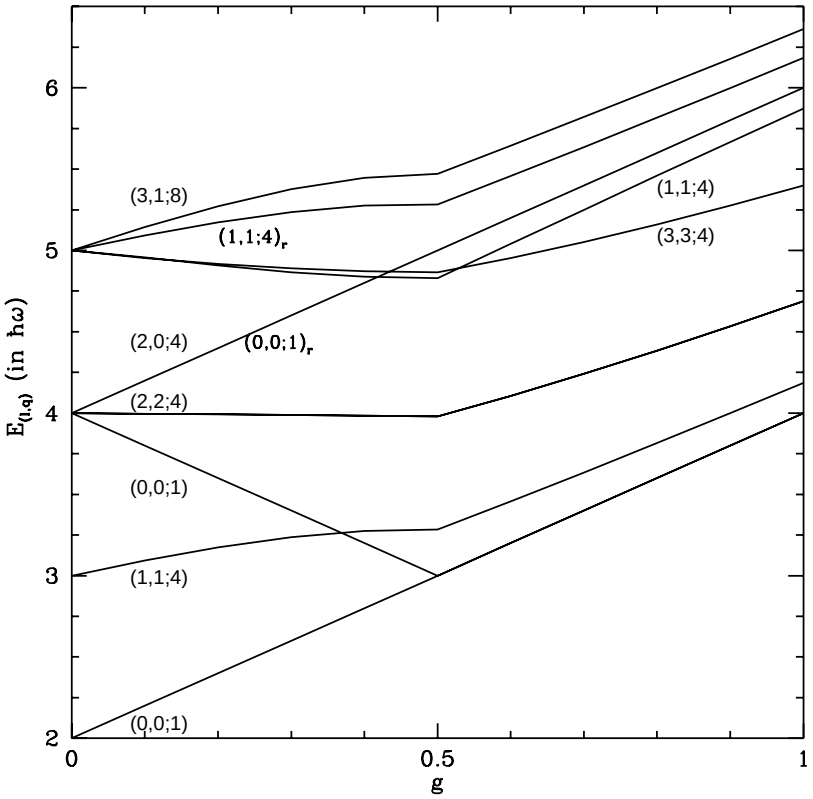

Figure 1. A spectrum of some low-energy states in the two-body problem as a function of the interaction parameter g. The states are labelled by $(l, q; D)$, where the first two entries denote the two angular quantum numbers and the last entry shows the degeneracy of the level. Further, the radial excitations are denoted by the subscript 'r'.

is fairly straightforward although cumbersome. We have truncated the basis such that the highest energy state in the basis is 100 $\hbar\omega$. The results for the low-lying states in the spectrum obtained through both methods are displayed in figure 1 (for $0 \leqslant g \leqslant 1$) and figure 2 (for $g \geqslant 1$). For $g < \frac{1}{2}$, it is more convenient to use the diagonalization procedure since the direct solution to equation (34) requires one to numerically fix two separate parameters, d and β. On the other hand, it is easier to solve the Heun equation (37) for $g \geqslant \frac{1}{2}$ since one only has to fix one parameter β. In general, we have used both methods to arrive at the spectrum of low-lying states. For small values of $g(< 1)$, however, the solutions of the differential equation produce eigenvalues which are somewhat smaller than the ones obtained by the diagonalization procedure. For large values of g, however, there is no perceptible difference between the results from the two methods.

Figure 1 shows the energies for some values of l and q. Each level is labelled by $(l, q; D)$, where D is the degeneracy of the level away from the noninteracting limit; the degeneracy is computed by counting the allowed values of $\pm l$ and $\pm q$ using the parity and supersymmetry transformations for a given level. A subscript 'r' on the label (l, q, D) denotes the radial excitation which is simply inferred from the existence of the towers. The bosonic ground state has the predicted behaviour for all g; it is linear with a gradient of 2 as a function of g. The corresponding wavefunction goes as $|x|^g$ as $x \to 0$. In contrast, the

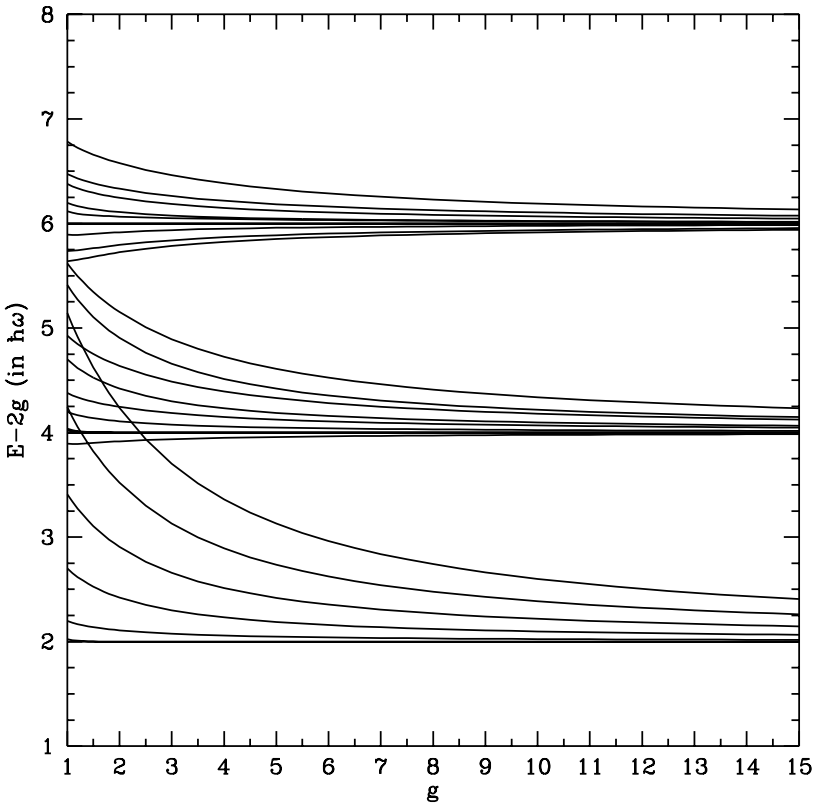

Figure 2. A spectrum of states as a function of g showing the strong coupling behaviour. We have shown $E - 2g$ instead of the energy itself.

level $(0, 0; 1)$ starting at $E = 4$ has an entirely different behaviour. It is exactly solvable for all g. According to the previous discussion, $\mathrm{d}E/\mathrm{d}g = -2$ for $g < \frac{1}{2}$ (with the wavefunction going as $|x|^{1-g}$ for small x) and $\mathrm{d}E/\mathrm{d}g = 2$ for $g > \frac{1}{2}$ (with the wavefunction going as $|x|^g$); thus $\mathrm{d}E/\mathrm{d}g$ is discontinuous at $g = \frac{1}{2}$.

At $g = 0$, the gradients $\mathrm{d}E/\mathrm{d}g$ for all the levels can be calculated using the first-order perturbation theory as shown in the next section. For *large* values g, we find that all the energies converge to $2g$ plus even integers as shown in figure 2. This amazing behaviour can be understood using the WKB method as shown in section 3.5.

3.4. Perturbation theory around $g = 0$

It is interesting to use the perturbation theory to calculate the changes in energy from $g = 0$, and to compare the results with the numerical analysis. We will only describe the first-order perturbation theory here, the example we will consider is the fermionic ground state which is doubly degenerate for $N = 2$.

In general, the naive perturbation theory fails at $g = 0$ because most $g = 0$ eigenstates do not vanish as $X_{ij} \to 0$; hence the expectation value of $1/X_{ij}^2$ diverges. This problem can be tackled by using a special kind of perturbation theory which was first devised for anyons [28–30]. We will first describe the idea for N particles and then devote our

attention to $N = 2$. Instead of solving the equation $H\Psi = E\Psi$, where H is given in equation (3), we perform a similarity transformation to $\tilde{H} = X_N^{-g} H X_N^{g}$ and $\tilde{\Psi} = X_N^{-g}\Psi$, where $X_N \equiv \prod_{i<j} |X_{ij}|^g$. We then find that $\tilde{H} = H_0 + \tilde{V}$, where H_0 is the noninteracting Hamiltonian (with $g_1 = g_2 = 0$), and

$$\tilde{V} = g \sum_{i \neq j} \frac{1}{X_{ij}} \left(x_j \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial x_i} \right). \tag{47}$$

The first-order changes in energy may now be obtained by calculating the expectation values (or matrix elements, in the case of degenerate states) of $\tilde{V}$ to the zeroth-order (noninteracting) eigenstates. These expectation values can be shown to be convergent for all states. Note that $\tilde{V}$ only contains two-body terms. Although $\tilde{V}$ is not Hermitian, it is guaranteed that its expectation values are real because the original problem has a Hermitian Hamiltonian H.

For $N = 2$, we find that

$$\tilde{V} = -2g \left(\frac{1}{R} \frac{\partial}{\partial R} + \frac{\cot 2\theta}{R^2} \frac{\partial}{\partial \theta} \right) \tag{48}$$

in hyperspherical coordinates. Note that $\tilde{V}$ commutes with both L and Q, so that we only have to consider its matrix elements within a particular block labelled by the eigenvalues l and q. Let us now use (48) to compute the first-order change in the states which have $E = 3$ at $g = 0$. There are four such states, with $l = \pm 1$ and $q = \pm 1$ (labelled $(1, 1; 4)$ in figure 1); two of these states are actually the ground states of the two-fermion system. Due to parity and SUSY, these four states remain degenerate for all g. Hence, it is sufficient to calculate the first-order change in the state with, say, $(l, q) = (1, 1)$. Since this state is unique at $g = 0$, we only need to use the nondegenerate perturbation theory with $\tilde{V}$. The normalized wavefunction for this state is

$$\Psi = \frac{1}{\pi\sqrt{2}} (\cos\theta - \sin\theta) \exp[\mathrm{i}(\phi + \psi)]\ R \exp[-R^2/2]. \tag{49}$$

We now obtain the expectation value

$$\int_0^\infty R^3\, \mathrm{d}R \int_{-\pi/4}^{\pi/4} \cos(2\theta)\, \mathrm{d}\theta \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \int_{-\pi}^{\pi} \mathrm{d}\psi\ \Psi^* \tilde{V} \Psi = g. \tag{50}$$

We can see from figure 1 that this gives the correct first-order expression for the energy $E = 3 + g$ near $g = 0$ for the states labelled $(1, 1; 4)$; their first radial excitations $(1, 1; 4)_r$ therefore have $E = 5 + g$.

We can similarly calculate the first-order expressions for the energies near $g = 0$ for all the other levels shown in figure 1. We find that $E = 2 + 2g$ for the lower state labelled $(0, 0; 1)$ (i.e. the bosonic ground state); $E = 4 + 2g$ for its radial excitation $(0, 0; 1)_r$, and the states $(2, 0; 4)$; $E = 4 - 2g$ for the upper state $(0, 0; 1)$; $E = 4$ for the states $(2, 2; 4)$; $E = 5 - g/2$ for the states $(1, 1; 4)$ and $(3, 3; 4)$; and $E = 5 + 3g/2$ for the states $(3, 1; 8)$.

3.5. Large-g perturbation theory

We can study the solutions of equation (37) for large values of g by using an expansion in $1/g$. For any value of b and c, we will only study the lowest energy, E, and we will calculate the leading-order terms in E and the wavefunction $\Theta(x)$. We first note that the terms of the order of g^2 in (37) can be satisfied only if $E = 2g + \mathrm{O}(1)$. Next, we assume

that E and Θ have WKB expansions [31] of the form

$$E = 2g + 2b + 2c + 2 + f_0 + \frac{f_1}{g} + \mathrm{O}(1/g^2)$$
$$\Theta = \exp\left[w_0(x) + \frac{w_1(x)}{g}\right]. \tag{51}$$

The boundary condition $\Theta = 1$ implies that $w_0(0) = w_1(0) = 0$. To the order of g, equation (37) gives the first-order differential equation

$$(1 - x^2)\frac{\mathrm{d}w_0}{\mathrm{d}x} = b - c - \frac{f_0}{2}x. \tag{52}$$

We now look at solutions which are nonzero only for $x \geqslant 0$. We demand that Θ should neither diverge nor vanish (since the lowest energy solution should be nodeless) anywhere in the range $0 \leqslant x \leqslant 1$. Hence, the functions w_0 and w_1 should not diverge to ∞ nor $-\infty$ in that range. This fixes $f_0 = 2(b - c)$, so that

$$E = 2g + 4b + 2 = 2g + |l + q| + 2 \qquad \text{and} \qquad \Theta = (1 + x)^{b-c}. \tag{53}$$

Similarly, there are solutions which are nonzero only for $x \leqslant 0$. These have

$$E = 2g + 4c + 2 = 2g + |l - q| + 2 \qquad \text{and} \qquad \Theta = (1 - x)^{c-b}. \tag{54}$$

We now go to the order of 1 in equation (37). For solutions which are nonzero only for $x \geqslant 0$, we find that

$$E = 2g + 4b + 2 + \frac{c^2 - b^2}{g} = 2g + |l + q| + 2 - \frac{lq}{4g}$$
$$\Theta = (1 + x)^{b-c} \exp\left[\frac{(b - c)(b + c + 2)}{2g}\left(\ln(1 + x) - \frac{x}{1 + x}\right)\right]. \tag{55}$$

We can similarly find solutions which are nonzero only for $x \leqslant 0$, by changing $x \to -x$ and interchanging $b \leftrightarrow c$, i.e. $l + q \to l - q$ and $lq \to -lq$, in equation (55).

We see from figure 2 that these formulae correctly describe the leading behaviour of E. In fact, the large-g behaviour is already visible in figure 1, for some states, as we approach $g = 1$. The various levels shown in that figure have the following WKB energies; $E = 2g + 2$ for both the $(0, 0; 1)$ states (one of these is the bosonic ground state and the other is the fermionic ground state for $g > \frac{1}{2}$ as discussed later); $E = 2g + 2 + \frac{1}{4}g$ for the states $(1, 1; 4)$; $E = 2g + 2 + 1/g$ for the states $(2, 2; 4)$; $E = 2g + 2 + \frac{9}{4}g$ for the states $(3, 3; 4)$; $E = 2g + 4 - \frac{1}{4}g$ for the states $(1, 1; 4)$; $E = 2g + 4$ for the radial excitation $(0, 0; 1)_r$ and the states $(2, 0; 4)$; $E = 2g + 4 + \frac{1}{4}g$ for the radial excitations $(1, 1; 4)_r$; and $E = 2g + 4 + \frac{3}{4}g$ for the states $(3, 1; 8)$. We have also checked that the leading-order wavefunctions in equations (55) agree remarkably well with the correct wavefunctions $\Theta(x)$ obtained by solving the Heun equation (37) even if g is not very large.

It is easy to see from equations (53) and (54) that for large g, the ground state and also the excited states become infinitely degenerate. This is so because one can choose the quantum numbers l and q in infinitely many ways such that the energies are the same as $g \to \infty$. Further, the spacings now become twice the spacing at $g = 0$ since l and q have the same parity mod 2.

The large-g behaviour therefore displays a remarkable similarity to the problem of a particle in a uniform magnetic field where the Landau level spacing is twice the cyclotron frequency, and each level is infinitely degenerate.

3.6. Fermionic ground-state energy

The fermionic ground-state energy has a very unusual behaviour as can be seen from figure 1. For $0 < g < 0.367$, the ground-state energy monotonically and nonlinearly increases from 3 to 3.266 along the curve labelled $(1, 1; 4)$. Beyond this point, for $0.367 < g < 0.5$, the ground-state energy monotonically and linearly decreases from 3.266 to 3 along the upper curve $(0, 0; 1)$ satisfying $E = 4 - 2g$. For $g \geqslant \frac{1}{2}$, the fermionic and bosonic ground-state energies are identical and are given by the curve $(0, 0; 1)$ which satisfies $E_0 = 2 + 2g$, i.e. both the ground states monotonically increase with g. Thus, the fermionic ground state consists of three pieces as a function of g, while the bosonic ground state is given by the single line $E_0 = 2 + 2g$ for all $g \geqslant 0$.

For two particles, one can understand why the fermionic and bosonic ground-state energies are identical for $g > \frac{1}{2}$ as follows. In this range of g, the ultraviolet potential near $x = 0$ is infinite and it prevents tunnelling between the regions $x > 0$ and $x < 0$ (see the appendix). For two identical particles, an exchange necessarily takes us from a region with $x > 0$ to a region with $x < 0$ according to (25). If tunnelling between the two regions is forbidden, it becomes impossible to compare the phases of the wavefunction of a given configuration of the two particles with the wavefunction of the exchanged configuration. Thus, it is impossible to distinguish bosons from fermions if $g > \frac{1}{2}$, and their energy levels must be identical.

It is possible that the same argument will go through for more than two particles; however, we need to understand the ultraviolet regularization of the three-body interactions properly in order to prove that rigorously. If the argument holds, then we would have the interesting result that the N-fermion ground-state energy is also given by (7) for $g > \frac{1}{2}$, while it may show one or more level crossings for $g < \frac{1}{2}$.

4. Discussion and summary

To summarize, we have studied a two-dimensional Hamiltonian whose eigenstates have a novel two-particle correlation. We have shown the existence of several classes of exact solutions in the many-body problem. We have analysed the two-particle problem in detail and have shown that it is completely solvable by reducing it to an ordinary differential equation in one variable which can be solved exactly for a subset of states and numerically otherwise. The two-body problem is integrable since there are four constants of motion in involution. We have also discussed the perturbation theory for both small and large coupling strengths. In the strong interaction limit, the system simplifies and bears a remarkable resemblance to the Landau level structure.

We have also clarified in the appendix the ultraviolet prescription which is required to make sense of an inverse-square (singular) potential especially at small coupling strengths. In particular, we emphasize that it is, in general, not sufficient to specify that the wavefunctions are regular and square integrable to obtain an energy spectrum uniquely when dealing with singular interactions. In some domains of the coupling strength, we also need to specify the ultraviolet regularization to make complete sense of the results. We do this by demanding that as the parameter $g \to 0$, the energy levels should smoothly approach the known noninteracting levels. We believe that this discussion is quite general and may have a wider applicability to Hamiltonians with singular interactions.

Interesting problems for the future would be to extend this analysis to more than two particles, and to find an application of our model to some physical system. Recently, we have come to know that our model has been generalized to three (and higher) dimensions

with novel three-body (and many-body) correlations [32].

Acknowledgments

Four of us (RKB, JL, MVNM and DS) would like to acknowledge financial support from NSERC (Canada). RKB would like to acknowledge the hospitality at the Institute of Physics, Bhubaneswar where part of this work was done. MVNM and DS thank the Department of Physics and Astronomy, McMaster University for hospitality.

Appendix

We begin directly from equation (34). Given the real number $g \geqslant 0$ satisfying $g_1 = g(g-1)$, $P(x)$ could go, as $|x| \to 0$, as either $|x|^g$ or $|x|^{1-g}$ or even as a general superposition of the two powers. We therefore need to define the problem more carefully in order to pick out a desired solution [14].

As mentioned above in the text, we demand the following. First, the limit $g = 0$ should give all the noninteracting two-particle solutions, both bosonic and fermionic. Secondly, all wavefunctions and energies, E, should be *continuous* functions of g, but the first derivative of E need not be continuous (indeed $\mathrm{d}E/\mathrm{d}g$ is not always continuous at $g = \frac{1}{2}$ as we saw earlier). Finally, for $g > 1$, the wavefunction should go as $|x|^g$, and not as $|x|^{1-g}$ which diverges at $x = 0$.

From these three requirements, it is clear that for $g \geqslant \frac{1}{2}$, the wavefunctions must go purely as $|x|^g$, whereas for $g < \frac{1}{2}$, the wavefunction could go either as $|x|^g$ or $|x|^{1-g}$ or a superposition of the two.

We will now show that we can satisfy the above requirements if we redefine the problem with a different potential in an *ultraviolet* region $|x| < x_0$. We take the potential to be

$$\begin{aligned} V(x) &= \frac{g(g-1)}{x^2} \qquad \text{for } |x| > x_0 \\ &= \frac{u^2}{x_0^2} \qquad \text{for } |x| < x_0 \end{aligned} \tag{A.1}$$

where $u \tanh u = g$ if $0 \leqslant g < \frac{1}{2}$ and $u = \infty$ if $g \geqslant \frac{1}{2}$. Eventually, of course, we have to take the limit $x_0 \to 0$ to recover our original problem. Note that the potential in the ultraviolet region is not symmetric under $g \to 1-g$ for $g \leqslant 1$. Hence, the energy spectrum does not have this symmetry.

To see why equations (A.1) work, we note that the wavefunction, for $|x|$ slightly greater than x_0 (where x_0 is much smaller than any physical length scales, such as the width of the harmonic oscillator potential), is generally given by

$$\begin{aligned} P(x) &= x^g + d_+ x^{1-g} \qquad \text{if } x > x_0 \\ P(x) &= (-x)^g + d_- \, (-x)^{1-g} \qquad \text{if } x < -x_0. \end{aligned} \tag{A.2}$$

(For the exceptional case $g = \frac{1}{2}$, we have to replace $|x|^g$ and $|x|^{1-g}$ by $|x|^{1/2}$ and $|x|^{1/2} \ln |x|$ respectively.)

Now consider the first case in (A.1), i.e. $0 \leqslant g < \frac{1}{2}$. Since the energy, E, is much less than the potential in the inside region $|x| < x_0$ (this is necessarily true for any finite value of E as $x_0 \to 0$), the wavefunction in that region is given by

$$P(x) \simeq \cosh\left[\left(\frac{u}{x_0} + \mathrm{O}(x_0)\right) x + \delta\right] \tag{A.3}$$

where δ can be a complex number, and the term of $O(x_0)$ arises from the energy, E, which is much less than $(u/x_0)^2$. We now match the wavefunction and its first derivative, or, more simply, the ratio $P'(x)/P(x)$ at $x = x_0 \pm \epsilon$ and at $x = -x_0 \pm \epsilon$, where ϵ is an infinitesimal number. We then find three possibilities.

(i) The wavefunction may be even about $x = 0$. Then $\delta = 0$, and $d_+ = d_-$ must vanish as x_0^{1+2g} as $x_0 \to 0$. (The behaviour of $d_\pm$ can be deduced by equating the terms of $O(x_0^{-1})$ and $O(x_0)$ in P'/P at $x = x_0\pm$.) In the limit $x_0 \to 0$, therefore, the wavefunction goes purely as $|x|^g$.

(ii) The wavefunction may be odd about $x = 0$. Then $\delta = i\pi/2$, and $d_+ = d_-$ must diverge as x_0^{2g-1} as $x_0 \to 0$. The wavefunction is proportional to $\mathrm{sgn}(x)\ |x|^{1-g}$ in that limit.

(iii) In the general asymmetric case, we find that we must have δ of the order of x_0^{1-2g}, and $d_+ = -d_- = d$ of O(1). (This is found by equating terms of $O(x_0^{-1})$ and $O(x_0^{-2g})$ in P'/P at $x = \pm x_0$.) The wavefunction is therefore a superposition of the form

$$P(x) = |x|^g + d\,\mathrm{sgn}(x)|x|^{1-g}. \tag{A.4}$$

The cases (i) and (ii) arise if either l or q is zero in equation (34), since the equation is invariant under $x \to -x$ in that case. This is precisely when $b = c$ and the equation is exactly solvable. We thus see that the even solutions go as $|x|^g$, while the odd solutions go as $|x|^{1-g}$. If neither l nor q is zero, i.e. $b \neq c$, we have case (iii) where a superposition of the two powers are required.

The second case in (A.1), i.e. $g \geqslant \frac{1}{2}$, is relatively simpler to analyse since the wavefunction must be zero in the inside region $|x| \leqslant x_0$. By imposing this condition on the wavefunction in the outside region, we see that both d_+ and d_- must vanish as $x_0 \to 0$. Hence, the wavefunction will go purely as $|x|^g$ in that limit. However, since there is no tunnelling possible through the infinite barrier separating $x > x_0$ from $x < -x_0$, we will generally have wavefunctions which are nonzero only for $x > x_0$ or only for $x < -x_0$. This is indeed true as we saw earlier for the solutions of the Heun equation for $g > \frac{1}{2}$.

We would like to emphasize that the relation $u \tanh u = g$ in equation (A.1) is absolutely essential in order to have the possibility of $P(x) \sim |x|^g$ for $g < \frac{1}{2}$. If u were to take any other value, we would find that $P(x)$ necessarily goes as $|x|^{1-g}$ in the limit $x_0 \to 0$. A similar fine tuning of u is necessary in the CSM for $g < \frac{1}{2}$. Incidentally, the strongly repulsive potential in the ultraviolet region explains the peculiar result that the bosonic ground-state energy increases monotonically with g even though the potential away from the ultraviolet region becomes more and more attractive as g goes from 0 to $\frac{1}{2}$. One can show from equation (A.1) that the integrated potential $\int_{-1}^{1} 1\,\mathrm{d}x\ V(x)$ is actually *positive* and large if x_0 is small, and it increases as g varies from 0 to $\frac{1}{2}$.

Several comments are in order at this stage.

(i) A similar fine tuning is also required in the CSM if $g < \frac{1}{2}$ is to be allowed, as has been done by several people [15–17]. Historically, both Calogero [1] and Sutherland [2] restricted themselves to $g > \frac{1}{2}$. In a sense they could do that since the free fermion limit corresponds to $g = 1$; thereby they avoided the problem with $g < \frac{1}{2}$. However, one cannot reach the free bosonic limit smoothly in that case. Both of these authors believed $g < \frac{1}{2}$ to be unphysical because they chose a particular regularization. What we have argued here is that one can choose an alternative regularization (called Sutherland's resonance condition) which allows one to go continuously all the way upto $g = 0$ and hence reach the free bosonic limit continuously. We have not seen this being clearly stated in CSM literature before, although Scarf [14] discusses this issue in a different problem containing the inverse-square potential.

It is worth noting that CSM has only two-body interactions. Hence, the entire discussion

here is also valid in the many-body case in CSM. This is in contrast to our problem where, for $N > 2$, one also has to analyse the ultraviolet regularization of the three-body interactions.

(ii) One important consequence of our regularization is that many of the states have discontinuities in $\mathrm{d}E/\mathrm{d}g$ at $g = \frac{1}{2}$; the fermionic ground state also has a level crossing at $g = 0.367$. Further, for each value of $g_1 < 0$, there are two possible ground states since the ultraviolet regularization depends on g and not on g_1.

(iii) In other words, our Hamiltonian, H, has several self-adjoint extensions (SAE) for each value of g_1. What we have done is to choose a particular SAE for $g_1 > 0$ and two different SAE for $g_1 < 0$. As a result, we have found that for every value of g_1 in the range $-\frac{1}{4} < g_1 < 0$, there are two possible ground-state energies since, as seen above, the SAE depends on g rather than on g_1. Actually, there is an even more general SAE possible for any value of g_1 where another real parameter (besides g) has to be introduced; however we shall not discuss that here.

References

[1] Calogero F 1969 *J. Math. Phys.* **10** 2191
Calogero F 1969 *J. Math. Phys.* **10** 2197
[2] Sutherland B 1971 *J. Math. Phys.* **12** 246
Sutherland B 1971 *J. Math. Phys.* **12** 251
Sutherland B 1971 *Phys. Rev.* A **4** 2019
[3] Sutherland B 1972 *Phys. Rev.* A **5** 1372
[4] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
[5] Shastry B S 1988 *Phys. Rev. Lett.* **60** 639
[6] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1395
[7] Trugman S A and Kivelson S 1985 *Phys. Rev.* B **31** 5280
[8] Laughlin R B 1989 *Ann. Phys.* **191** 163
[9] Leinaas J M and Myrheim J 1977 *Nuovo Cimento* B **37** 1
[10] Wilczek F 1982 *Phys. Rev. Lett.* **49** 957
[11] Wu Y-S 1984 *Phys. Rev. Lett.* **53** 111
[12] Date G and Murthy M V N 1993 *Phys. Rev.* A **48** 105
[13] Murthy M V N, Bhaduri R K and Sen D 1996 *Phys. Rev. Lett.* **76** 4103
[14] Scarf F L 1958 *Phys. Rev.* **112** 1137
[15] Murthy M V N and Shankar R 1994 *Phys. Rev. Lett.* **73** 3331
[16] Isakov S B 1994 *Int. J. Mod. Phys.* A **9** 2563
[17] Bernard D and Wu Y-S 1995 *Proc. 6th Nankai Workshop on New Developments of Integrable Systems and Long-ranged Interaction Models* ed M L Ge and Y-S Wu (Singapore: World Scientific)
[18] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703
[19] Khare A and McCabe J 1991 *Phys. Lett.* B **269** 330
[20] Murthy M V N, Law J, Brack M and Bhaduri R K 1991 *Phys. Rev. Lett.* **67** 1817
[21] Sporre M, Verbaarschot J J M and Zahed I 1991 *Phys. Rev. Lett.* **67** 1813
[22] Landau L D and Lifshitz E M 1958 *Quantum Mechanics* (Reading, MA: Addison-Wesley) appendix D, p 496
[23] Kilpatrick J E and Larsen S Y 1987 *Few Body Syst.* **3** 75
[24] Sen D 1992 *Phys. Rev. Lett.* **68** 2977
Sen D 1992 *Phys. Rev.* D **46** 1846
[25] Erdelyi A (ed) 1955 *Higher Transcendental Functions, Bateman manuscript project* vol III (New York: McGraw-Hill) p 57
[26] Dutra A de S 1993 *Phys. Rev.* A **47** R2435
[27] Dutt R, Khare A and Varshni Y P 1995 *J. Phys. A: Math. Gen.* **28** L107
[28] Sen D 1991 *Nucl. Phys.* B **360** 397
[29] McCabe J and Ouvry S 1991 *Phys. Lett.* B **260** 113
[30] Comtet A, McCabe J and Ouvry S 1991*Phys. Lett.* B **260** 372
[31] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods For Scientists and Engineers* (New York: McGraw-Hill)
[32] Ghosh P K 1996 *Novel Correlations in Arbitrary Dimensions (Preprint)* cond-mat/9607009